NUMERICAL MODELING OF GRAVITATIONAL WAVES IN THE OCEAN BY THE DOUBLE-LAYER POTENTIAL METHOD

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The problem of wave structure formation on a water surface for a prescribed initial disturbance is considered. A method of modeling gravitational waves is suggested in which the velocity potential of the liquid is sought in the form of a double-layer potential. The method suggested is tested by solving the problem of wave propagation on shallow water. Calculations of formation of a wave train from the initial disturbance are made in the case of periodic motion in a two-dimensional model (the problem of Stokes waves). The results of the calculations have demonstrated the high effectiveness of the suggested method in solving problems in which it is necessary to determine a wave profile with good accuracy.

Analysis of wave propagation on water is a classical problem [1], and interest in it is dictated by the need to answer particular questions in solving a variety of applied problems.

The primary physical factors that determine the existence of waves on water are the gravitational forces and the surface tension [2]. In the present work we are concerned with gravitational waves.

The main conditions that exert an influence on wave propagation are the relations between the wavelength, amplitude, and depth of the liquid. They correspond to deep and shallow water. In the phenomenon of wave propagation these notions have a relative meaning and are compared to the wavelength. Deep water means bodies of water where the depth exceeds the wavelength, and shallow water means ones where it is less than the wavelength. Hence it follows that for a complex linear wave consisting of a number of sine waves with substantially different wavelengths a body of water can turn out to be both deep and shallow at the same time. In good agreement with observations, theory shows that water can be considered to be deep when the depth of the body of water exceeds approximately half the wavelength but water is shallow if the depth of the body of water is approximately ten times smaller than the wavelength.

The character of wave propagation on deep and shallow water is fundamentally different, which is attributable to the fact that wave motion quickly attenuates with increasing depth. The attenuation follows an exponential law, and already at a depth equal to half the wavelength the amplitude of displacement of water particles in the vertical direction is a factor of 23 smaller than on the water surface, and at a depth equal to a whole wavelength it is even more than 500 times smaller [3]. As a result, when the depth of a body of water is larger than a wavelength, the disturbance already scarcely reaches the bottom. In the case of shallow water, the entire body of liquid from the surface to the bottom is involved in the disturbance.

Rather short-lived sources of emergence of waves can serve as factors of excitation of long waves. These are earthquakes, eruptions of coastal and underwater volcanos, and underwater blasts. In the ocean, waves generated by an underwater shock often have a great length and are propagated with a high velocity. These waves, tsunami, often carry huge energy and when the waves break down on coasts they cause disastrous destruction [4].

Under initial conditions of a certain type tsunami evidently appears in the form of a solitary wave (a soliton) rather than in the form of a wave train. A soliton on water was observed by Russell in 1834 during barge tests on a canal. Stokes and Airy, contemporaries of Russell, regarded negatively the results of observations of this wave, but later Boussinesq (1872) and Rayleigh (1876) confirmed the possibility of Russell's observation of a solitary

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wave on water. In 1895 Korteveg and de Vries derived an equation based on the "shallow water" approximation [2] that corresponded to the Russell experiments.

Numerous works in which wave processes on water are modeled use, in essence, the "shallow water" approximation, i.e., an approximation in which the wave amplitude is much less than the depth of the body of water ($\varepsilon = a/h \ll 1$) and the wavelength is much greater than the depth ($\delta = h/l \ll 1$).

The present work suggests a calculation method and provides results of calculation of wave structure formation from an initial disturbance that are not based on the "shallow water" approximation.

Formulation of the Problem. Let an initial disturbance for which the "shallow water" approximations are violated at the initial moment be created in a body of water. Assume that at $t = 0$ all the characteristics of the disturbed region are known, in particular, the surface profile is known. In this model, the bottom relief is assumed to be horizontal [5-7].

This two-dimensional problem will be considered in a Cartesian coordinate system (the x axis is along the horizontal, the z axis is along the vertical). The problem is to determine the evolution in time of the liquid surface profile with allowance for gravitational forces.

Since water is practically incompressible, the continuity equation can be written in the form

$$\operatorname{div} \vec{u} = 0 . \tag{1}$$

By assuming that the liquid motion is a vortex-free flow, we can consider one scalar function $\vec{u} = \nabla \varphi(x, z, t)$ instead of the two components of the vector function $\vec{u}$ and one equation for the potential

$$\Delta\varphi = 0 . \tag{2}$$

instead of two equations for the velocity components.

The equation of motion in the gravitational field has the form

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla)\,\vec{u} + \frac{1}{\rho_0}\nabla P = -g\vec{j} . \tag{3}$$

Equation (3) can be integrated over z and represented as

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{P - P_0}{\rho_0} + gz = 0 . \tag{4}$$

If we solve the Laplace equation for the velocity potential, then we can, in principle, find the pressure in the liquid from Eq. (4).

We now dwell on the boundary conditions of the problem under consideration. The equation of motion of the upper part of the boundary is of the form

$$\frac{dz^s}{dt} = (\vec{u})_z = \frac{\partial\varphi}{\partial z} . \tag{5}$$

After a standard substitution we write condition (5) as

$$\frac{\partial z}{\partial t} + u_x \frac{\partial z}{\partial x} = u_z$$

or

$$\frac{\partial z}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial\varphi}{\partial z} . \tag{6}$$

Assume that on the lower part of the boundary the following condition is fulfilled:

$$\frac{\partial \varphi^{s}}{\partial n} = \frac{\partial \varphi^{s}}{\partial z} = 0 . \tag{7}$$

On the right-hand side of the boundary we assume the condition

$$\varphi^{s} = 0 , \tag{8}$$

i.e., we will consider the disturbances not to have yet reached the right-hand boundary at the considered moments of time. At the point $x = 0$ we use the following conditions of symmetry:

$$\frac{\partial \varphi^{s}}{\partial n} = - \frac{\partial \varphi^{s}}{\partial x} = 0 . \tag{9}$$

On the upper movable part of the boundary we take the condition of pressure continuity across the water–air interface as the boundary condition, thus neglecting the action of the surface tension forces. With account for this we obtain from (4) a boundary condition in the form

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^{2} + g z^{s} = 0 . \tag{10}$$

Assuming that at the initial moment the wave profile and the liquid velocity are known, we have the relations

$$z^{s}(x) = Z^{0}(x) , \quad u_{x}(x) = 0 , \quad u_{z}(x) = 0 . \tag{11}$$

Method of Numerical Solution of the Problem. The suggested method of numerical solution of the problem formulated is a combination of the method of potentials and the method of straight lines. Its essence is as follows. The nodes of the spatial grid are situated only on the boundary of the region. The velocity potential is sought in the form of a double-layer potential that is symmetric in the variable x

$$\varphi(M) = \iint\limits_{\frac{1}{2}\Omega} \gamma(P_{s}) \frac{\partial}{\partial n_{P_{s}}} \left[\ln \left(\frac{1}{r_{MP_{s}}} \right) + \ln \left(\frac{1}{\overset{*}{r}_{MP_{s}}} \right) \right] d\sigma_{P_{s}} , \tag{12}$$

for determination of the density of which on the upper boundary use is made of the finite-difference approximation of the corresponding Fredholm integral equation of the second kind, which reflects the fact that the double-layer potential undergoes a discontinuity across the boundary:

$$\iint\limits_{\frac{1}{2}\Omega} \gamma(P_{s}) \frac{\partial}{\partial n_{P_{s}}} \left[\ln \left(\frac{1}{r_{M_{s}P_{s}}} \right) + \ln \left(\frac{1}{\overset{*}{r}_{M_{s}P_{s}}} \right) \right] d\sigma_{P_{s}} - \varphi(M_{s}) = - \pi \gamma(M_{s}) .$$

On the right-hand and lower boundaries use is made of the finite-difference approximation of the corresponding equations, which are obtained by exact differentiation of the formula that approximates the velocity potential. In the partial differential equation for the z components of the wave profile the derivatives with respect to the variable x are approximated by finite differences, and the velocities u_x and u_z are calculated by exact differentiation of the formula that approximates the potential of velocities. The time derivatives are not approximated. As a result, the initial problem reduces to the Cauchy problem for a nonlinear system consisting of ordinary differential equations in the variable t and steady-state equations for the values of the density function of the double-layer potential at the nodes of the surface grid.

To solve this Cauchy problem, a STIFF package of programs was used that was modified by the present authors, which permitted its use for solving the Cauchy problem for a combined system of differential and steady-state equations. The main convenience of this approach is an automatic procedure for selecting the time step in the STIFF program in accordance with the prescribed local accuracy of integration.

The unknown quantities in the problem are: a) the density of the double-layer potential at the grid nodes on the surface; b) the z coordinates of the wave profile; c) the velocity potential at the grid nodes on the surface.

In constructing the grid, the corresponding number of nodes on the wave profile and on the right-hand and lower boundaries is prescribed.

On the wave profile the grid was constructed as follows. The length of the curve of the initial profile was calculated and divided by the prescribed number of nodes. Next, the nodes were spaced on the curve the same distance from each other. The projections of the nodal points on the X axis form a nonuniform grid in the variable x, which was then used in the numerical modeling. This method allows satisfactory approximation of even almost vertical initial wave profiles (for instance, 50 points are enough to approximate the initial profile of a water column with a height of 5 km at a width of only 50 m; the length of the region of solution of the problem is 10 km).

On the right-hand boundary a quasi-uniform grid in the variable z was constructed that was compressed toward the water surface so that the steps in x and z at the upper right corner were equal to each other. On the lower boundary a uniform grid in the variable x was used.

Modeling of Stokes Waves. Consider the problem of modeling Stokes waves on water of arbitrary depth in the approximation of a two-dimensional model. A zone of depth H is considered to be the domain of solution of this problem:

$$-\infty < x < \infty\,; \quad -H < z < Z\,(x, t)\,. \tag{13}$$

We will seek $Z(x, t)$ in the class of functions once continuously differentiable with respect to x and t that possess the additional property of periodicity in the variable x:

$$Z\,(x, t) = Z\,(x + L, t)\,. \tag{14}$$

In the domain (13) it is necessary to solve the Laplace equation for the potential of velocities

$$\Delta\varphi = 0\,. \tag{15}$$

The velocity vector $\vec{u}$ was expressed in terms of the potential as $\vec{u}(x, z, t) = \nabla\varphi(x, z, t)$.
At $Z = -H$ the condition of equality of the z-component of the velocity to zero was set

$$u_z\,(x, z = -H, t) = \frac{\partial\varphi}{\partial z}\,(x, z = -H, t) = 0\,, \quad -\infty < x < \infty\,; \quad 0 < t < \infty\,. \tag{16}$$

On the water surface the kinematic conditions

$$\frac{\partial Z}{\partial t}\,(x, t) + u_x\,(x, z = Z, t)\,\frac{\partial Z}{\partial x}\,(x, t) = u_z\,(x, z = Z, t) \tag{17}$$

and the dynamic condition

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\,|\nabla\varphi|^2 + gZ = 0 \tag{18}$$

were written.

It is known that Stokes waves are unstable. The profile and the potential of velocities for them were prescribed in the form of the following functions:

$$Z\,(x, t) = A\cos\,(kx - w\,(k)\,t) + \frac{1}{2}\,kA^2\cos\,(2\,(kx - w\,(k)\,t))\,, \tag{19}$$

$$\varphi\,(x, z = Z\,(x, t)\,, t) = \frac{w\,(k)\,A}{k}\exp\,(kz)\sin\,(kx - w\,(k)\,t)\,. \tag{20}$$

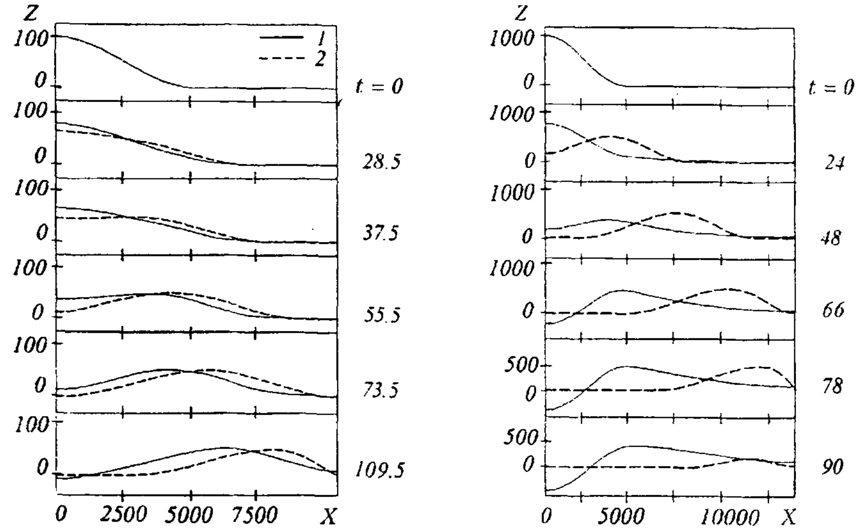

Fig. 1. Time evolution of wave profiles: 1) calculation without using the "shallow water" approximation; 2) calculation in the "shallow water" approximation. Z, X, m; t, sec.

Here

$$w(k) = \sqrt{gk}\left(1 + \frac{1}{2}(kA)^2\right).$$

In modeling Stokes waves the initial conditions were prescribed by formulas (19), (20) at $t = 0$. Relations (13)-(18) and (19) with account for expression (20) completely determine the mathematical formulation of the problem of modeling Stokes waves on a body of water of arbitrary depth in the two-dimensional model. This problem is the unsteady-state Dirichlet problem for the Laplace equation in a zone with a movable upper boundary.

The potential of velocities is sought in the form of a double-layer potential with a dipole density function that is periodic in x on the upper part of the boundary:

$$\varphi(M, t) = \int_{P^{\text{up}}} \gamma^{\text{up}}(P) \frac{\cos\theta_{MP}}{r_{MP}} dl_P + \int_{P^{\text{down}}} \gamma^{\text{down}}(P) \frac{\cos\theta_{MP}}{r_{MP}} dl_P. \tag{21}$$

The periodicity condition (14) necessitates a search for a dipole density on the upper part of the boundary in the form of the function

$$\gamma^{\text{up}}(x_P, Z(x_P, t)) = \gamma^{\text{up}}(x_P + L, Z(x_P + L, t)). \tag{22}$$

which is periodic in the variable x. In this connection, the first integral in formula (21) is represented as a sum of individual integrals over segments of the length L. The number of terms in the sum can be estimated from the condition $\max(r_{MP}) >> L$.

Thus, on the upper and lower parts of the boundary the function $\gamma^{\text{up}}(P)$ is sought on the segment $[0, L]$. In solving the problem of modeling Stokes waves the grid was constructed in the same way as in solving the problem of modeling other gravitational waves on water.

Calculated Results and Discussion. The calculation program composed was tested on solution of the problem of wave propagation on shallow water.

On the left in Fig. 1 results of wave profile calculations for initial data corresponding to a ratio of the water depth to the wavelength of 1/5 are given at different times. The initial profile has a height of 100 m, a water depth of 2000 m, and a wavelength of 10,000 m. The grid contained 101 nodes on the wave profile.

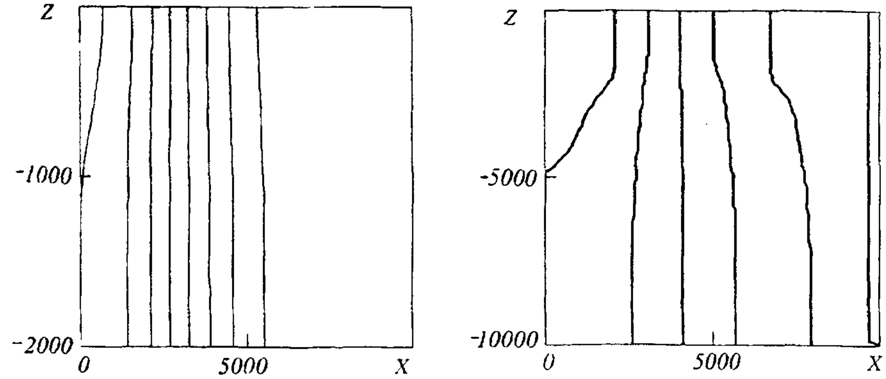

Fig. 2. Level lines of the velocity potential for the profiles given in Fig. 1.

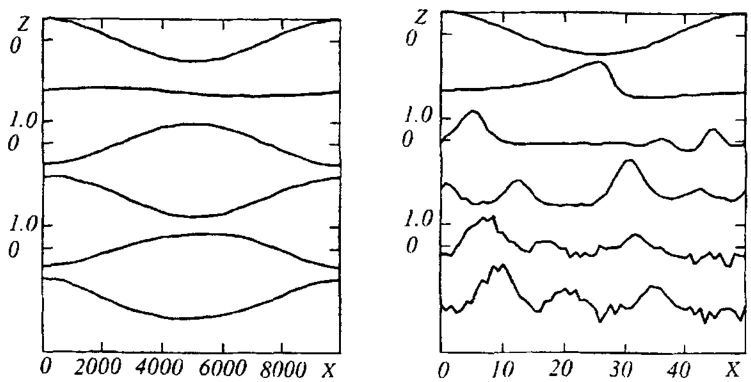

Fig. 3. Calculated results for periodic gravitational waves on water for initial data corresponding to the linear (at the left) and nonlinear (at the right) cases.

On the right in Fig. 1 results of wave profile calculations for initial data corresponding to the limit of applicability of the "shallow water" approximation where the ratio of wavelength to water depth ratio is equal to unity are presented at different times. The initial profile has a height of 1 km, a water depth of 10 km, and a wavelength of 10 km. The grid contained 31 nodes on the wave profile. In performing this calculation the time step was chosen automatically to provide a local relative accuracy of integration with respect to time of 0.01 and was, on the average, 2 sec. In this calculation the Jacobi matrix was subjected to factorization only four times.

Figure 2 shows level lines of the potential of velocities corresponding to the wave profiles represented in Fig. 1 for a time equal to 78 sec.

Figures 1 and 2 indicate that at a ratio of water depth to wavelength equal to 1 the "shallow water" approximation begins to be violated. The equipotentials are no longer vertical lines, which necessitates account for the dependences of the velocities on the variable z.

The calculations have demonstrated that the wave obtained from calculations in the "shallow water" approximation runs faster than that obtained without using this approximation. Moreover, the stronger the violation of the "shallow water" approximation, the stronger the discrepancy in the results. This is attributed to the neglect of the z component of the velocity in the "shallow water" approximation.

On the left in Fig. 3 results of calculations for a Stokes wave are given for the following initial data: the water depth is 100 m, the wavelength is 10 km, the wave height is 1 m. Wave profiles are shown at different times (the curves from the top down, respectively): t = 3.203; 112.1; 237.011; 474.022; 720.642; 871.176. The grid contained 31 nodes on the wave profile. It is seen that for the prescribed initial conditions (this is practically the linear case) the wave form remains almost unchanged with time.

On the right in Fig. 3 calculated results for a Stokes wave are given for the following initial data: the water depth is 2 m, the wavelength is 50 m, the wave height is 1 m. Wave profiles are shown at different times (the curves from the top down, respectively): $t = 0.337$; 4.045; 10.112; 14.83; 19.549; 20.223. The grid contained 61 nodes on the wave profile. These initial data correspond to the case where nonlinearity is of great importance.

As is seen in Fig. 3, in the linear case the wave motion is stable, while in the nonlinear case, the Stokes wave turns into a train of waves with time.

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NOTATION

ε, δ, dimensionless small parameters; a, wave amplitude; h, depth of the body of water; l, wavelength; t, time variable; x, z, space variables; $\vec{u}$, velocity vector of the liquid; $\varphi(x, z, t)$, velocity potential; ρ_0, liquid density; $P(x, z, t)$, pressure; g, free-fall acceleration; $\vec{j}$, unit vector in the direction of the gravitational force; P_0, ambient pressure on the liquid surface; $P_s = (x^s, z^s, t)$, points of the boundary of the liquid volume; d/dt, substantial derivative; $\partial/\partial t$, $\partial/\partial z$, $\partial/\partial x$, partial derivatives with respect to the variables t, z, x, respectively; $Z(x, t)$, unknown upper movable part of the domain boundary; u_x, u_z, velocity vector components along the x and z axes; $\varphi^s(x^s, z^s, t)$, velocity potential on the boundary; $Z^0(x)$, wave profile at the initial time; $M(x, z)$, arbitrary point in the domain; Ω, round solid angle; $\gamma(P_s)$, density of the double-layer potential at points of the liquid boundary; $\partial/\partial n$, derivative with respect to the normal to the surface at the point P; r_{MP}, distance between the points M and P; r^*_{MP}, distance equal to r_{MP} which is symmetrical relative the fluid surface; $d\sigma_P$, differential of the domain of the solution; M_s, points of the boundary of the liquid volume; L, period of the function $Z(x, T)$; k, wave number; $P^{up}(x, Z(z, t))$, $P^{down}(x, -H)$, points on the upper and lower parts of the boundary of domain (13); $\gamma^{up}(P)$, $\gamma^{down}(P)$, dipole densities on the upper and lower parts of the boundary; $\cos\theta_{MP}$, angle between the positive direction of the normal to the surface at the point P and the segment PM; dl_P, differential of the arc length.

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